Section 3.7 Notes

Rational Functions

1 Introduction

**Definition**

- A *rational function* is fraction of two polynomials.

- For example, $f(x) = \dfrac{2x^2 - 1}{3x^2 + 2x - 5}$ is a rational function.

Properties of Rational Graphs

- The graph of every rational function is *smooth* (no sharp corners)

- However, many rational functions are *discontinuous*. In other words, the graph might have several separate "sections" to the graph, similar to the way piecewise functions look.

  For example, remember the graph of $f(x) = \frac{1}{x}$:
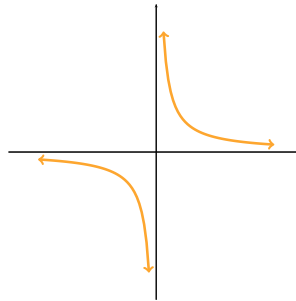
  Notice how the graph is drawn in two separate pieces.
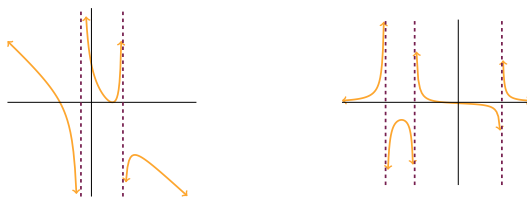
Asymptotes

- An *asymptote* to a function is a line which the function gets closer and closer to without touching.

- Rational functions have two categories of asymptote:

  1. vertical asymptotes
  2. asymptotes which determine the end behavior - these could be either horizontal asymptotes or slant asymptotes
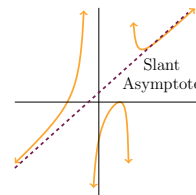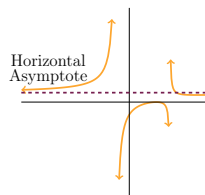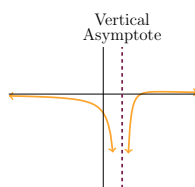
2 Vertical Asymptotes

Description

- A *vertical asymptote* of a rational function is a vertical line which the graph never crosses, but does get closer and closer to.

- Rational functions can have any number of vertical asymptotes

- The number of vertical asymptotes determines the number of "pieces" the graph has. Since the graph will never cross any vertical asymptotes, there will be separate pieces between and on the sides of all the vertical asymptotes.

Finding Vertical Asymptotes

1. Factor the denominator.

2. Set each factor equal to zero and solve.

   The locations of the vertical asymptotes are nothing more than the x-values where the function is undefined.

Behavior Near Vertical Asymptotes

The multiplicity of the vertical asymptote determines the behavior of the graph near the asymptote:

Multiplicity	Behavior
even	The two sides of the asymptote match - they both go up or both go down.
odd	The two sides of the asymptote don't match - one side goes up and the other goes down.

3 Zeros of Rational Functions

Finding Zeros of a Rational Function

- Technically, to find the zeros you plug in zero for y and solve for x. However, with rational equations this will always simplify in the same way:

$$f(x) = \frac{(x-2)(x+1)}{(x-6)(x+3)}$$

$$(x-6)(x+3)\cdot 0 = \frac{(x-2)(x+1)}{\cancel{(x-6)(x+3)}}\cdot \cancel{(x-6)(x+3)}$$
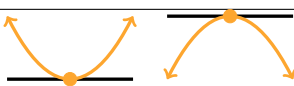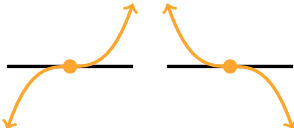
$$0 = (x-2)(x+1)$$

$$x-2=0 \text{ or } x+1=0$$

$$x=2 \text{ or } x=-1$$

- You can skip these steps and just set the factors in the numerator equal to zero and solve.

Behavior Near Zeros

The behavior of a rational function near its zeros works exactly like polynomial functions:
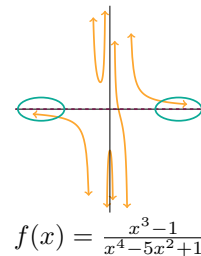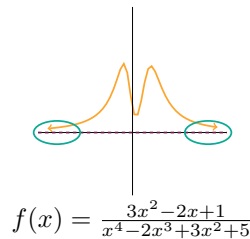
Multiplicity	Behavior
even	The graph bounces off the x-axis at the zero.
odd	The graph crosses through the x-axis at the zero.

4 End Behavior

Case 1: Degree on Top is Less Than the Bottom

The polynomial should look like $\dfrac{ax^m + \cdots + c}{bx^n + \cdots + d}$, where $m < n$.

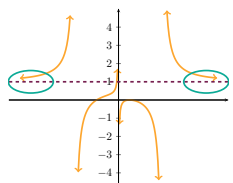- In this case, the graph has a horizontal asymptote along the x-axis ($y = 0$).

- For example:

$$f(x) = \frac{3x^2 - 2x + 1}{x^4 - 2x^3 + 3x^2 + 5}$$

$$f(x) = \frac{x^3 - 1}{x^4 - 5x^2 + 1}$$

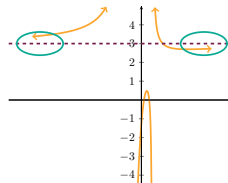**Case 2: Degree on Top is Equal to the Bottom**

The polynomial should look like $\dfrac{ax^n + \cdots + c}{bx^n + \cdots + d}$.

- In this case, the graph has a horizontal asymptote along $y = \frac{a}{b}$.

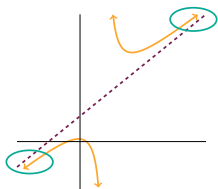- For example:

$$f(x) = \frac{x^3 - 3x + 2}{x^3 - 16x}$$
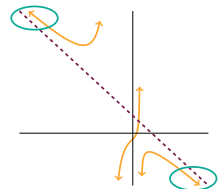
$$f(x) = \frac{3x^4 - 4x^3 + 16x^2 - 20x + 5}{x^4 - 4}$$

Case 3: Degree on Top is Exactly One Greater Than the Bottom

The polynomial should look like $\dfrac{ax^{n+1} + \cdots + c}{bx^n + \cdots + d}$.

- In this case, the graph has a slant asymptote along some line $y = mx + b$.

- To find the equation of the line, perform polynomial long division - the quotient gives you $mx + b$ (the remainder can be ignored).

- For example:

$$f(x) = \frac{x^2 + x - 2}{x - 4}$$
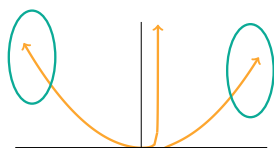
$$f(x) = \frac{-2x^3 + x^2 - 10x + 5}{x^2 + 2x - 3}$$

Case 4: Degree on Top is Two or More than the Bottom

The polynomial should look like $\dfrac{ax^m + \cdots + c}{bx^n + \cdots + d}$, where $m - n \geq 2$.
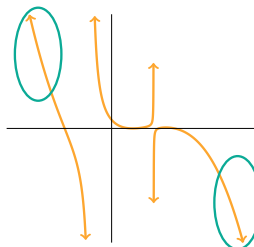
- In this case, the end behavior is *not asymptotic* - it behaves exactly like polynomial end behavior.

- Simplify $\frac{ax^m}{bx^n} = \frac{a}{b}x^{m-n}$

	coefficient is positive	coefficient is negative
exponent is even		
exponent is odd		

- For example:

$$f(x) = \frac{x^3 - 3x^2 + 1}{x - 2}$$

$$f(x) = \frac{x^5 - 5x^4 + 2x^3 + 22x^2 - 35x + 15}{-x^2 + x + 2}$$

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5   Graphing Rational Functions

Goal

When we draw our graph, we want to have three features correct:

1. The end behavior

2. The behavior near the roots

3. The behavior near the vertical asymptotes

   Other details might be slightly incorrect, but we only need those three aspects right. For a better graph, you'd need to plug in a large number of points.

Method

1. Use the factored form of the numerator to find and plot the zeros.

2. Use the factored form of the denominator to find and plot the vertical asymptotes.

3. Determine the end behavior and plot any horizontal/slant asymptotes.

4. Plug in and plot any extra points you want - if you have a horizontal asymptote along the x-axis, you *must* have at least one extra point. In this case, you'll probably want two extra points on either side of all the zeros and vertical asymptotes.

5. Draw in each piece of the graph separated by vertical asymptotes, starting from the left-most section and finishing with the right-most. Draw each piece left-to-right as well.

   - On the left-most section:
     - Draw in the left-side end behavior.
     - Cross/bounce at the x-axis for each zero in that section
     - Once you've finished all the zeros, you need to go up or down towards the vertical asymptote. Go up if you're currently above the x-axis and go down if you're currently below the x-axis.
   - In each middle section:
     - Go up or down towards the vertical asymptote on the left, based on the multiplicity of that asymptote.
     - Cross/bounce at the x-axis for each zero in that section.
     - Go up or down towards the vertical asymptote on the right. Go up if you're currently above the x-axis and go down if you're currently below the x-axis.

   - In the right-most section:
     - Go up or down towards the vertical asymptote, based on the multiplicity of that asymptote.
     - Cross/bounce at the x-axis for each zero in that section.
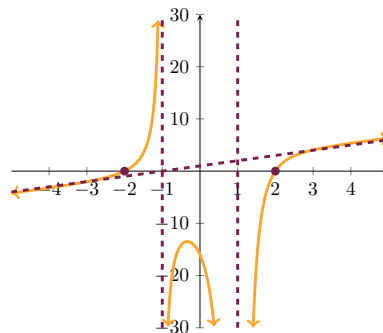   - Draw in the right-side end behavior.
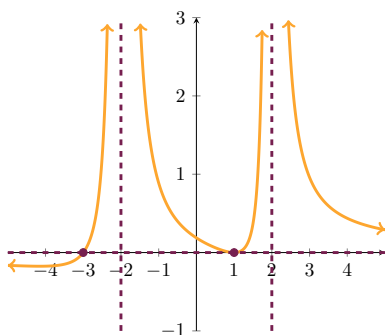
Examples
Graph the following functions:
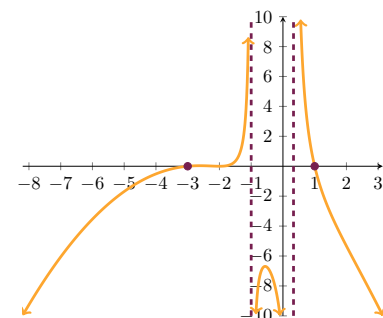
1. $f(x) = \dfrac{-3x^2 + 6x - 3}{x^2 + x}$
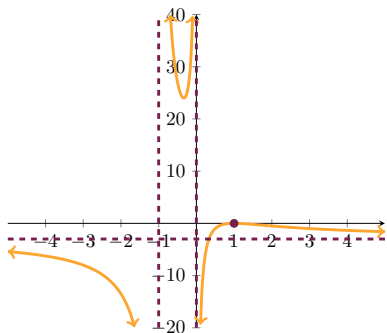
2. $f(x) = \dfrac{x^4 - 16}{x^3 - x^2 - x + 1}$

3. $f(x) = \dfrac{(x-1)^2(x+3)}{(x-2)^2(x+2)^2}$

4. $f(x) = \dfrac{(x+2)^2(x-1)(x+3)}{(-3x+1)(x+1)}$

Special Case: $f(x) = \dfrac{ax + b}{cx + d}$

Rational functions of this form can also be graphed using transformations of the graph of $y = \frac{1}{x}$.
For example, consider $y = \frac{x+3}{2x-2}$.

1. Perform polynomial long division:
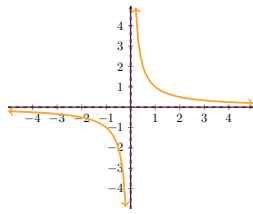
$$\frac{x+3}{2x-2} = \frac{1}{2} + \frac{4}{2x-2}$$

2. In the denominator of the fraction, factor out any coefficient of x.

$$\frac{1}{2} + \frac{4}{2x-2} = \frac{1}{2} + \frac{4}{2(x-1)}$$
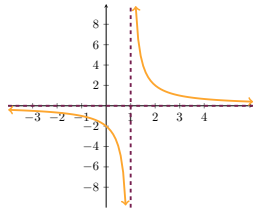
3. Pull out the number on top and bottom of the fraction:

$$\frac{1}{2} + \frac{4}{2(x-1)} = \frac{1}{2} + \frac{4}{2} \cdot \frac{1}{x-1} = \frac{1}{2} + 2 \cdot \frac{1}{x-1}$$
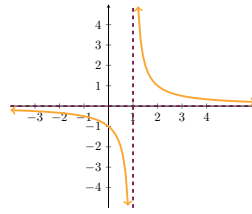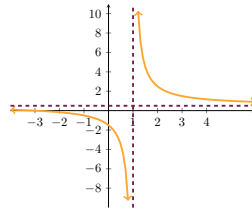
4. Graph using transformations.

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$$y = \frac{1}{x}$$

$$y = \frac{1}{x-1}$$

$$y = 2 \cdot \frac{1}{x-1}$$

$$y = 2 \cdot \frac{1}{x-1} + \frac{1}{2}$$